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A Remark on Nondecidabilities of Initial Value Problems of ODEs

Günter Hotz
Fakultät für Mathematik und Informatik
Universität des Saarlandes

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[1]Anschrift: D-66041 Saarbrücken
Emailadresse: hotz@cs.uni-sb.de
URL:www-hotz.cs.uni-sb.de

The Problem

We consider the problem to decide if a system of ordinary differential equations

$$\dot{z}(t) = f(z(t)), \quad z(t) := (z_1(t), ..., z_n(t))$$
$$f \quad \text{given by a program on a} \quad \mathbf{R} - \text{machine}$$

has a solution for given inititial conditions

$$z(0) := (\zeta_1, ..., \zeta_n) \in \mathbf{R}^n$$

over the intervall $[0, b)$ with $b \in \mathbf{R}$.

1 A Special Solution

A finite **R**-machine is a modell of a computer based on the existence of a finite number m of registers, which are variables over **R**. In future when speaking about $\mathbf{R}-$machines we allways mean finite ones. We allways include a program on the machine when speaking of a **R**-machine. This makes the machine to a deterministic device. The machine offers for a finite set O of indices $j \in O$ a set of operations

$$x_i := \tau_j(x_l, x_k) \quad \text{for} \quad i, k, l \in \{1, ..., m\} \tag{1}$$

The set of operations usually contains the arithmetic operations $(+, -, *, /)$ and conditional operations to determine the sequence of operations to be applied depending from the actual content of the registers. Assuming the machine to be deterministic means that the available operations are partial mappings

$$(x_{t+1}, j_{t+1}) := \tau(x_t, j_t) \quad \text{for} \quad x_{t+1}, x_t \in \mathbf{R}^m \quad j_{t+1}, j_t \in O, \quad t \in N.$$

The assumption that the machine is deterministic means that for each state there exists maximal one operation, which is applicable to a given state of the machine. We define the sequence

$$(x_0, 0), ..., (x_{t+1}, j_{t+1}) := \tau(x_t, j_t), ... \tag{2}$$

to be the computation of our machine, if the operations for all steps are well defined.
Some of the operations may produce states on which none of the available

operations is applicable.In this case the computation stops. An example for this is the division by zero. But there are other situations to be considered. The programmer may have forgotten to specify for some cases how the computation has to branch or he wishes the computation to stop in a certin state. It is well known that for machines of this type there does not exist any general procedure to decide, if programs terminate or go on in the computation for ever.
We define a sytem of ODEs to interpolate the seqence (2). We do this by connecting the neighbours of points $z_t := (x_t, j_t) \in \mathbf{R}^{m+1}$ generatet by the computation (1) by edges and describe these polygons $z(t)$, which are continuous curves as far they are defined and they are differentiable in the intervalls $(t, t+1]$. The polygons which interpolate the computations obviously are solutions of the differenial equation system

$$\dot{z}(t) = \tau(z(\lfloor t \rfloor)) - z(\lfloor t \rfloor) \quad \text{for} \quad \lfloor t \rfloor \leq t < \lfloor t \rfloor + 1 \tag{3}$$

uniquily determined by the given initial value $z(0) := (x_0, 0)$. This means that we have a procedure to decide the halting problem for our program, if there exists a decidable necessary and sufficient condition for the existence of a soluion of the initial value problem of ODEs over the infinite intervall. This means that the following theorem holds.

Theorem 1 *There does'nt exists an universal procedure to decide if the intial value problem of a system of ODEs with the right side beeing computable on* $\mathbf{R}-$*machines, has a solution over the interval* $[0, \infty)$

Tis problem is frequenty called the mortality problem. The right sides of the ODEs we constructed are not continous and this could be the reason that the theorem holds. The following section shows that this is not a property, which is essential for this result.

2 ODEs with Differentiable Right Sides

As stated before it holds for the definition areas $Q(\tau_i)$ of the operations τ_i

$$Q(\tau_i) \bigcap Q(\tau_l) = \emptyset \quad \text{for all} \quad i, l \in O.$$

Therefore we are able to describe the set (1) of operations by a single partial computable mapping

$$S : \mathbf{R}^n \to \mathbf{R}^n$$

. We assume that we have an $\mathbf{R}-$machine which computes $S(x)$ for $x \in \mathbf{R}^n$ in a time unit if S is defined for x. We interpolate the points of the sequence

$$x_0, x_1, ..., x_{r+1} := S(x_r), ...$$

by a sequence of inductively computable polynomials, such that for the curve $x : [0, T] \to \mathbf{R^n}$ we constructed the k first derivations do exist in each point $x(t)$ for $0 \leq t \leq T$.
We define for each $r \in \mathbf{N}$ n polynomials

$$\begin{aligned} \mathcal{P}_r &:= (p_{1,r}, p_{2,r}, ..., p_{n,r}) \\ p_{i,r}(t) &:= a_{i,r,k} * t^k + ... + a_{i,r,0} \quad \text{for} \quad i \in \{1, ..., n\} \end{aligned}$$

which satisfy the following conditions

$$\begin{aligned} \mathcal{P}_0(t) &:= x_0 + t * (x_1 - x_0), \\ \mathcal{P}_r(0) &:= x_r \quad \text{for} \quad r \in \mathbf{N}, \\ \mathcal{P}_r(1) &:= x_{r+1} \quad \text{for} \quad r \in \mathbf{N}, \\ \frac{d^l}{dt^l}\mathcal{P}_{r+1}(0) &:= \frac{d^l}{dt^l}\mathcal{P}_r(1) \quad \text{for} \quad r \in \mathbf{N}, \quad l \in \{1, ..., k\}. \end{aligned}$$

These conditions define the polynomial systems uniquely because $\mathcal{P}_0$ is defined and assuming $\mathcal{P}_r$ beeing defined it follows that the system $\mathcal{P}_{r+1}$ is defined uniquely because the derivations of degree $0, 1, ..., k$ of the polynomials $\mathcal{P}_{r+1}(t)$ of degree k for $t := 1$ are given by $\frac{d^l}{dt^l}\mathcal{P}_r(1)$ for $l \in \{0, ..., k\}$.
We define now an ODEs

$$\dot{x} = \frac{d}{dt}F(x, t)$$

by defining for $x_0 \in \mathbf{R}^r$

$$F(x_0, t) := \mathcal{P}_{\lfloor t \rfloor}(t - \lfloor t \rfloor)$$

$x(t) := F(x_0, t)$ is the uniquely determined solution of the defined differential equation with initional condition x_0. Obviously F is $\mathbf{R}$-computable and k-times differentiable. This construction works for each $\mathbf{R}$-machine and each program on this machine. The program does not stop if and only if $F(x_0, t)$ is defined for the whole intervall $[o, \infty)$. $\mathbf{R}$-machines are universal if they can use 3 or more $\mathbf{R}$-registers. Therefore the following theorem holds.

Theorem 2 *For each $k \geq 3, k \in \mathbf{N}$ there exists an ODEs with $k-$times differentiable and $\mathbf{R}$-computable right side for which no universal algorithm on a $\mathbf{R}-$machine exists to decide if a given initial condition for this ODEs has a solution over the intervall $[0, \infty)$*

If we measure each iteration step of S not by a unit of time but shorten these steps geometrically as $c * 2^{-n}$ then we get the more general

Theorem 3 *There exists ODEs with k-time differentiable right sides, for which no universal algorithm exists to decide if a given initial condition has a solution over a given interval* $[0, c)$.

It is wellknown that it is not generally decidable, if a proceedure of a program for given initial conditions during the computation will by applied. This means, that it is undecidable if a given operator of (1) will be applied for given initial conditions. We generalize our interpolation by allowing for different states a different degree k of differentiability. It follows directly the

Theorem 4 *It is generally undecidable if for the solution of an initial condition problem for ODEs there exists for a given interval a solution, which is* $l-$*times,* $l \in \mathbf{N}, l > 0$ *differentiable.*

The results show that we may consider ODEs as infintesimal machines and that there exist universal machines of this type an that all the undecidability results of discrete machines may find their equivalent in the infinitesimal case.

3 Literatur

Relationships between discrete and continuous dynamical systems have been studied since the time of Euler. We use discrete algoritms to approximate the solutions of differential equations and to prove the existence of solutions of ODEs under certain conditions. C. Moore [5] raises the question of decidability of stability questions of dynamical systems. The paper "Analog computation with continuous ODEs" of M. S. Brannicky [6] is motivated by the question if analog computers are as universal as digital computers. He shows that Turing machines and special cases of Turing machines can be simulated on solutions of ODEs which fullfills a Lippschitz condition. He mentions the relevance of this result for decidability questions but does not report a non decidability result. V.D. Blondel,.. [7]prove that the mortality and convergence of peacewise affine discrete dynamical systems is undecidable. This result corresponds to our theorem 1.
Our approach is based on $\mathbf{R}$-machines, which are idealisations of Turing machines. This machines have been introduced under this name by Blum,..[8]. We studied the problem of the stability of solutions initial value problems in [2], [3], [4] in connection with infinte computations with an convergend output - we called this configuration analytical machine - and proved some

hierarchie theorems and proved that the stability of this continuous dynamical systems is not decidable even if we allow analytical computations. The halting problem of Turing machines is decidable with analytical machines.

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